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COMMENT

The potential for a homogeneous spheroid in a spheroidal coordinate system: II. At an interior point

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Abstract. The internal potential for a homogeneous spheroid in a rectangular coordinate system may be obtained with the substitution of the root of the equation for confocal spheroids being zero in the expression of the external potential. In a spheroidal coordinate system, the integration of the internal potential would be composed of two parts: inner and outer volumes with respect to the interior point of a spheroid, which generates the explicit forms of the internal potential and the expressions on spheroidal surface, where the external and internal potentials should merge, for both prolate and oblate spheroids. The reductions for the potential at the centre of a spheroid to a sphere and the numerical results compared between the rectangular and spheroidal coordinate systems have certainly confirmed the correctness of the formulae of the internal potentials for both spheroids.

1. Introduction

Given the formula of the potential at the exterior point for a homogeneous spheroid in a rectangular coordinate system (Kellogg 1929, Hopfner 1933, MacMillan 1958), the potential at the interior point can be calculated with the substitution of $s=2a$ in (1a) and (1b) (in which a is semimajor axis) or of $u=0$ in (2a) and (2b) or of $k=0$ in (4a) and (4b) (Wang 1988).

Hobson (1931) obtained the potentials for homogeneous prolate and oblate spheroids in spheroidal coordinate systems (η, θ, ϕ). The relations between the rectangular and his spheroidal coordinate systems are as follows:

$$\left.\begin{aligned} x &= c \sinh \eta \sin \theta \cos \phi \qquad (1a)\\ y &= c \sinh \eta \sin \theta \sin \phi \qquad (1b)\\ z &= c \cosh \eta \cos \theta \qquad (1c)\end{aligned}\right\} \text{ for prolate system}$$

$$\left.\begin{aligned} x &= c \cosh \eta \sin \theta \cos \phi \qquad (2a)\\ y &= c \cosh \eta \sin \theta \sin \phi \qquad (2b)\\ z &= c \sinh \eta \cos \theta \qquad (2c)\end{aligned}\right\} \text{ for oblate system}$$

where c is the semifocal distance, η, θ, and ϕ are the spheroidal coordinates. His formulae for the internal and external potentials for both prolate and oblate spheroids are extremely complicated: the functions $Y_n^m(\theta, \phi)$ are not spherical harmonics in the common sense and the function $f(\theta, \phi)$ should be theoretically sought from a certain integral equation. Thus these formulae are impractical in the calculation; furthermore, the formulation depends upon the unnecessary rotational coordinate ϕ though the final outcomes might be independent of it.

Following the procedures performed in a previous paper (Wang 1988), we will derive the internal potentials for both prolate and oblate spheroids in their own coordinate systems.

2. The internal potential for a homogeneous spheroid in a spheroidal coordinate system

First let us introduce the following spheroidal coordinates which are related to the rectangular ones by

$$x = l(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\cos\phi \tag{3a}$$

$$y = l(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\sin\phi \tag{3b}$$

$$z = l\xi\eta. \tag{3c}$$

At the interior point of a spheroid, the integration of the potential in a spheroidal coordinate system breaks down into two parts: inner and outer volumes with respect to the interior point. Hence the expressions for the reciprocal of the distance between the spheroidal volume element and the interior point would have the two different forms (Wang 1988):

$$\frac{l}{r} = \sum_{n=0}^{\infty}(2n+1)P_n(\eta)P_n(\eta')Q_n(\xi)P_n(\xi')+\ldots \qquad \text{for } \xi>\xi' \tag{4a}$$

$$\frac{l}{r} = \sum_{n=0}^{\infty}(2n+1)P_n(\eta)P_n(\eta')P_n(\xi)Q_n(\xi')+\ldots \qquad \text{for } \xi<\xi' \tag{4b}$$

where ξ is the spheroidal radial coordinate of the interior point and ξ' is that of the spheroidal volume element; the notation is also true for spheroidal angular coordinates η, η' and azimuthal coordinates ϕ, ϕ'; P_n and Q_n are the Legendre functions of the first and second kinds. Because of rotational symmetry on ϕ, we have deleted the second parts in (4a) and (4b), which contain the separated function $\cos m(\phi-\phi')$, based on the fact that

$$\int_0^{2\pi}\cos m(\phi-\phi')\,\mathrm{d}\phi \equiv 0 \qquad \text{for } m\neq 0. \tag{5}$$

The two integrals of the internal potential transferred correspondingly from (4a) and (4b) are

$$V_1 = 2\pi l^2\sum_{n=0}^{\infty}(2n+1)P_n(\eta)Q_n(\xi) \times\int_1^{\xi}\int_{-1}^{+1}P_n(\xi')P_n(\eta')(\xi'^2-\eta'^2)\,\mathrm{d}\eta'\,\mathrm{d}\xi' \qquad \text{for } \xi>\xi' \tag{6a}$$

$$V_2 = 2\pi l^2\sum_{n=0}^{\infty}(2n+1)P_n(\eta)P_n(\xi) \times\int_{\xi}^{\xi_0}\int_{-1}^{+1}Q_n(\xi')P_n(\eta')(\xi'^2-\eta'^2)\,\mathrm{d}\eta'\,\mathrm{d}\xi' \qquad \text{for } \xi<\xi' \tag{6b}$$

where ξ_0 is the value of the spheroidal radial coordinate on the spheroidal surface. For the prolate spheroid, $\xi_0 = a/(a^2-c^2)^{1/2}$; for the oblate one, $\xi_0 = c/(a^2-c^2)^{1/2}$; where c is the semi-minor axis.

After mathematical simplification, the internal potential for a homogeneous prolate spheroid would be

$$V = V_1 + V_2 = \frac{M}{l}\left(Q_0(\xi_0)[1 - P_2(\xi)P_2(\eta)] + \frac{\xi(\xi^2-1)}{\xi_0(\xi_0^2-1)}(\tfrac{3}{2}\xi)P_2(\eta) + \frac{(\xi_0^2-\xi^2)}{\xi_0(\xi_0^2-1)}[\tfrac{1}{2} + P_2(\xi)P_2(\eta)]\right). \tag{7a}$$

The internal potential for a homogeneous oblate spheroid may be generated with the replacements of ξ by $\mathrm{i}\xi$, ξ_0 by $\mathrm{i}\xi_0$, and l by $-\mathrm{i}l$ in $(7a)$, namely,

$$V = \frac{\mathrm{i}M}{l}\left(Q_0(\mathrm{i}\xi_0)[1 - P_2(\mathrm{i}\xi)P_2(\eta)] + \frac{\xi(\xi^2+1)}{\xi_0(\xi_0^2+1)}(\tfrac{3}{2}\mathrm{i}\xi)P_2(\eta) + \frac{(\xi_0^2-\xi^2)}{(\mathrm{i}\xi_0)(\xi_0^2+1)}[\tfrac{1}{2} + P_2(\mathrm{i}\xi)P_2(\eta)]\right). \tag{7b}$$

On the spheroidal surface, both the internal and external potentials tend to the same values for both prolate and oblate spheroids, respectively, that is

$$V_s = \frac{M}{l}[Q_0(\xi_0) - Q_2(\xi_0)P_2(\eta)] \qquad \text{for a prolate spheroid} \tag{8a}$$

$$V_s = \frac{\mathrm{i}M}{l}[Q_0(\mathrm{i}\xi_0) - Q_2(\mathrm{i}\xi_0)P_2(\eta)] \qquad \text{for an oblate spheroid.} \tag{8b}$$

At the centres of prolate and oblate spheroids, the potentials become

$$V_0 = \frac{3}{2}\frac{M}{l}Q_0(\xi_0) \qquad \text{by setting } \xi = 1 \text{ and } \eta = 0 \text{ in } (7a) \text{ for a prolate spheroid} \tag{9a}$$

$$V_0 = \frac{3}{2}\frac{\mathrm{i}M}{l}Q_0(\mathrm{i}\xi_0) \qquad \text{by setting } \xi = 0 \text{ and } \eta = 1 \text{ in } (7b) \text{ for an oblate spheroid.} \tag{9b}$$

The forms $(9a)$ and $(9b)$ can be reduced to the spherical case if we let $l \to 0$:

$$\lim_{l\to 0}\frac{3}{2}\frac{M}{l}Q_0(\xi_0) = \lim_{l\to 0}\frac{3M}{2}\frac{Q_0(a/l)}{l} = \frac{3M}{4}\lim_{l\to 0}\frac{\mathrm{d}}{\mathrm{d}l}\left[\ln\frac{a+l}{a-l}\right] = \frac{3M}{2a} \tag{10a}$$

$$\lim_{l\to 0}\frac{3}{2}\frac{\mathrm{i}M}{l}Q_0(\mathrm{i}\xi_0) = \lim_{l\to 0}\frac{-3M}{2}\frac{(\tan^{-1}(c/l) - (\pi/2))}{l} = \frac{-3M}{2}\lim_{l\to 0}\frac{\mathrm{d}}{\mathrm{d}l}[\tan^{-1}(c/l)] = \frac{3M}{2c}. \tag{10b}$$

3. Numerical comparison between the rectangular and spheroidal coordinate systems

For a prolate spheroid, let $l = 25$ and $\xi_0 = \frac{5}{3}$; for the field point, let $\xi = \frac{41}{40}$ and $\eta = 0.6$. The corresponding values in the rectangular coordinate system are $x = 15.375$, $r = (y^2 + z^2)^{1/2} = 4.5$. Then the internal potentials in the two different coordinate systems turn out to be

$$V = 0.039\,237\,484\,351\ldots \qquad \text{from } (7a) \tag{11a}$$

and

$$V = 0.039\,237\,484\,351\ldots \qquad \text{from } (2a) \qquad \text{(Wang 1988).} \qquad (11b)$$

For an oblate spheroid, let $l = 25$ and $\xi_0 = \frac{4}{3}$; for the field point, $\xi = \frac{9}{40}$ and $\eta = 0.6$; that is $r = (x^2 + y^2)^{1/2} = 20.5$, $z = 3.375$. The potentials are

$$V = 0.035\,195\,465\,623\ldots \qquad \text{from } (7b) \qquad (12a)$$

and

$$V = 0.035\,195\,465\,623\ldots \qquad \text{from } (2b) \qquad \text{(Wang 1988).} \qquad (12b)$$

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